Empirical Tests of a Hedging Framework for the Quantum Binomial Options Pricing Model

Arnav Malhotra

July 2025

1 Abstract

The Quantum Binomial Options Pricing Model is a novel approach to pricing derivatives. It serves as a quantum analogue to the classical binomial options pricing model, using concepts from quantum mechanics like the Pauli matrices and Bloch sphere to represent the derivatives market. Through usage of a density matrix and mixed states, the model can potentially price derivatives more accurately. However, at this time there exist no clear formulae and methods for hedging derivatives through this model. This paper numerically tests the performance of the hedging framework corresponding to the model as compared to the classical model using a year of historical S&P 500 data. The simulation reveals that the Quantum Binomial Options Pricing model is, at least in its current state, not viable for usage in actually hedging derivatives. This is due to numerical instabilities, particularly in the Quantum Theta and Quantum Vega, the former of which ends up negative, betraying fundamental financial postulates, and the latter of which is blown out of scale due to the short term nature of the simulation. We conclude that the Quantum Binomial Options Pricing Model is currently unsuitable for practical applications and that our findings reveal the necessity of rigorously testing new financial models against benchmark models like the Classical Binomial Options Pricing Model.

2 Introduction

2.1 Hedging and the Greeks

In finance, a hedge is a position with the goal of offsetting potential losses from a variety of factors, like changes in a derivative's underlying asset price, changes in that asset's volatility, or a change in interest rates. There are many different strategies for hedging with derivatives, the most common being delta hedging, vega hedging, theta hedging, and rho hedging.

2.2 The Classical Binomial Options Pricing Model

The binomial options pricing model[1] (BOPM) is a method used to calculate the fair price of an option. It offers a discrete-time framework that maps out potential future price movements of an underlying asset, such as a stock, in a series of upward or downward steps. This "binomial tree" of possible prices allows for the valuation of the option at each point in time, working backward from the expiration date to the present. The formula for BOPM is what follows:

$$C_{t-\Delta t,i} = e^{-r\Delta t}\left(pC_{t,i} + (1-p)C_{t,i+1}\right) \tag{1}$$

Where:

- $C_{t,i}$ is the option's value at the i^{th} node at time t
- r is the risk-free rate
- p is the probability of an up move

2.3 Quantum Finance

Quantum finance is an emerging field concerned with applying theories in quantum mechanics to solve problems in finance. By leveraging the unique properties of quantum physics, it aims to create more accurate financial models and speed up calculations that are currently too complex for even the most powerful classical computers.

2.4 The Quantum Binomial Options Pricing Model

The quantum binomial options pricing model[3] (QBOPM), aims to serve as a quantum analogue to BOPM. Instead of assuming that an asset's price can only move up or down in discrete steps based on classical probabilities, QBOPM uses the principles of quantum mechanics to describe these potential price movements. This allows for a more nuanced and potentially more powerful way to analyze the evolution of asset prices.

In the BOPM, the market at any point is in a definite state (a specific price node). The QBOPM replaces this with a quantum state described by a density matrix, ρ. The density matrix is used in quantum mechanics to describe an ensemble of quantum states, and holds all the available information about a quantum system. For a two state system (analogous to the up and down moves in BOPM), ρ will be a 2×2 hermitian matrix, expressed as a linear combination of the identity matrix (I_2) and the three Pauli spin matrices, $\sigma_x, \sigma_y, \sigma_z$:

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} = \frac{1}{2}(I_2 + x\sigma_x + y\sigma_y + z\sigma_z) \tag{2}$$

where (x, y, z) is a real vector called the Bloch vector. To be a valid density matrix, ρ must satisfy these conditions:

$$\rho\,\mathrm{Tr}(\rho) = 1, \rho \succeq 0 \tag{3}$$

The Bloch vector defines a point in the Bloch sphere, where each point corresponds to a unique quantum state, allowing for the representation of superposition.

In QBOPM, the evolution of the market's quantum state is governed by a quantum operator, A. Like the density matrix, the operator A for a two-state system is a 2×2 matrix that can be constructed from the Pauli basis:

$$A = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = x_0 I_2 + x_1\sigma_x + x_2\sigma_y + x_3\sigma_z \tag{4}$$

While the market state and its evolution are described by matrices, any measurement or observation of the system must yield a classical, real-valued outcome. In quantum mechanics, the possible outcomes of a measurement of an observable (represented by an operator) are the eigenvalues of that operator. For QBOPM, the observable outcomes of the asset's growth over one period are the eigenvalues of the quantum operator A, which are denoted as a and b.

Eigenvalues are found by solving the characteristic equation, or $\det(A-\lambda I) = 0$, yielding

$$\lambda = x_0 \pm \sqrt{x_1^2 + x_2^2 + x_3^2} \tag{5}$$

These eigenvalues are the direct quantum analogue to the classical up and down states in the BOPM. The no-arbitrage condition is imposed in the QBOPM by requiring that the expected rate of return on the asset, when in the quantum state ρ and evolving according to operator A, must equal the risk-free rate, r. The quantum mechanical expectation value is given by the trace of the product of the density matrix and the operator:

$$1 + r = \mathrm{Tr}(\rho(I + A)) \implies \mathrm{Tr}(\rho A) = e^{r\Delta t} - 1 \tag{6}$$

Expanding this expression using the Pauli matrix representations for ρ and A leads to a constraint that defines the quantum equivalent of the risk-neutral world. This constraint can be solved for a "risk-neutral measure," M, which dictates the effective probabilities of observing the eigenvalues a and b in an arbitrage-free market. The resulting risk-neutral probabilities for the "up" state (b) and "down" state (a) are:

$$\begin{aligned} M_b &= \frac{e^{r\Delta t} - (1 + a)}{b - a} \\ M_a &= \frac{(1 + b) - e^{r\Delta t}}{b - a} \\ &= 1 - M_b \end{aligned} \tag{7}$$

With these components in place, the single-period QBOPM pricing formula for a European call option, C_Q, is constructed. It takes a form that is a direct analogue of the classical risk-neutral valuation formula, but uses the quantum-derived parameters:

$$C_Q = e^{-r\Delta t}[M_b h_b + M_a h_a] \tag{8}$$

where h_b and h_a are the option payoffs:

$$\begin{aligned} h_b &= [S_0(1+b) - K]^+ \\ h_a &= [S_0(1+a) - K]^+ \end{aligned} \tag{9}$$

However, even though the model was first introduced by Chen in 2010, no further research has been done into empirically testing the model and its corresponding hedging parameters against market data. This paper aims to fill that gap by deriving the analytical formulae for each of the QBOPM's main "Quantum Greeks" and creating a simulation based on historical market data to compare to the classical BOPM.

2.5 The Greeks

In financial risk management, the Greeks[2] represent the sensitivity of the value of a derivative to changes in underlying parameters, such as the price of the underlying asset, volatility of the underlying asset, or the passage of time. The Greeks are represented by the partial derivative of the derivative's value with respect to the parameter. Some of the most common first order Greeks are:

- Delta (Δ), the sensitivity of the derivative's value to changes in the price of the underlying asset, or $\Delta = \frac{\partial V}{\partial S}$
- Vega (ν), the sensitivity of the derivative's value to changes in the volatility of the underlying asset, or $\nu = \frac{\partial V}{\partial \sigma}$
- Theta (Θ), the sensitivity of the derivative's value to the passage of time, or $\Theta = -\frac{\partial V}{\partial \tau}$, $\tau = \Delta t$
- Rho (ρ), the sensitivity of the derivative's value to changes in the risk-free rate, or $\rho = \frac{\partial V}{\partial r}$
- Epsilon (ϵ), the sensitivity of the derivative (option)'s value to changes in the underlying dividend yield, or $\epsilon = -\frac{\partial V}{\partial q}$

There also exist second order Greeks, which are the rates of change of the first order Greeks. The most common is Gamma:

- Gamma (Γ), the sensitivity of Delta to changes in the underlying price, or $\Gamma = \frac{\partial^2 V}{\partial S^2}$

Many of these Greeks can be used to hedge to keep the portfolio neutral to changes in that Greek's underlying parameter, for example keeping a Delta neutral or Vega neutral portfolio.

3 Derivations of Quantum Greeks

3.1 Quantum Delta

Much like the classical Delta, the Quantum Delta, Δ_Q, is the first order partial derivative of C_Q with respect to S_0,

$$\Delta_Q = \frac{\partial C_Q}{\partial S_0} \tag{10}$$

This quantity represents the instantaneous rate of change of the option's value as the underlying stock price changes. In practical terms, it is the number of shares of the underlying asset an option writer must hold (or short) for each option sold to create a risk-neutral position.

$$\begin{aligned}
\frac{\partial C_Q}{\partial S_0} &= \frac{\partial}{\partial S_0} e^{-r\Delta t}[M_b h_b + M_a h_a] \\
&= \frac{\partial}{\partial S_0} e^{-r\Delta t}[M_b[S_0(1+b) - K]^+ + M_a[S_0(1+a) - K]^+] \\
&= e^{-r\Delta t}[M_b(1+b)\Theta(S_0(1+b) - K) + M_a(1+a)\Theta(S_0(1+a) - K)]
\end{aligned} \tag{11}$$

where $\Theta(x)$ is the Heaviside step function, which is defined as:

$$\Theta(x) := 1_{\mathbb{R}_{\geq 0}} \tag{12}$$

For an option that is deep in-the-money, such that it is guaranteed to be in-the-money in both future states, both $\Theta(S_0(1+b)-K)$ and $\Theta(S_0(1+a)-K)$ are equal to 1, and the formula simplifies to:

$$\Delta_Q^{ITM} = e^{-r\Delta t}[M_b(1+b) + M_a(1+a)] \tag{13}$$

A crucial observation arises from the formula for Δ_Q. It depends on the parameters $r, M_a, M_b, a,$ and b. The eigenvalues a and b are determined by the parameters (x_0, x_1, x_2, x_3) of A. Additionally, the risk-neutral condition $r = \mathrm{Tr}(\rho A)$ establishes a relationship between the parameters of A and the parameters of ρ, (x, y, z). Because of this relationship, Δ_Q is not just a function of volatility and interest rates like in the classical BOPM, but instead a function of the entire quantum state of the market, suggesting that Δ_Q can capture a finer-grained market structure for a more accurate hedging.

3.2 Quantum Gamma

While delta provides a first-order approximation of an option's price change, it is not constant. As the underlying asset's price moves, the delta changes. This second order effect is captured by Gamma, which measures the rate of change of delta. A portfolio with high gamma requires frequent rebalancing to maintain delta-neutrality.

Like the classical Gamma, Quantum Gamma, Γ_Q is the second order partial derivative of C_Q with respect to S_0, or the first order partial derivative of Δ_Q:

$$\Gamma_Q = \frac{\partial^2 C_Q}{\partial S_0^2} = \frac{\partial \Delta_Q}{\partial S_0} \tag{14}$$

$$\begin{aligned} \frac{\partial \Delta_Q}{\partial S_0} &= \frac{\partial}{\partial S_0} e^{-r\Delta t}[M_b(1+b)\Theta(S_0(1+b)-K) + M_a(1+a)\Theta(S_0(1+a)-K)] \\ &= e^{-r\Delta t}[M_b(1+b)^2\delta(S_0(1+b)-K) + M_a(1+a)^2\delta(S_0(1+a)-K)] \end{aligned} \tag{15}$$

Where $\delta(x)$ is the Dirac delta function, which is defined as:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \tag{16}$$

This is called the sifting property, and gives these two characteristics:

$$\delta(x) = 0, \text{ for } x \neq 0, \int_{-\infty}^{\infty} \delta(x)dx = 1 \tag{17}$$

Because of this usage of the Dirac Delta function, the financial interpretation of this Γ_Q formula is difficult. Γ_Q is 0 everywhere except at two specific stock prices, $S_0(1+b)-K$, and $S_0(1+a)-K$, where it is infinite. This is an artifact of the single period QBOPM, and would likely smooth out to a realistic and continuous function for a multi period model.

3.3 Quantum Vega

One of the most important factors in pricing an option is volatility. Therefore, it is important to be able to hedge against changes in it. This is where we can use Vega. Quantum Vega is defined as

$$\nu_Q = \frac{\partial C_Q}{\partial \sigma} \tag{18}$$

To derive ν_Q, first a relationship between volatility and the parameters of the quantum model must be defined. A natural quantum analogue to the classical volatility is the spread between eigenvalues, or $b-a$.

$$\begin{aligned} \sigma_Q = b - a &= (x_0 + \sqrt{x_1^2 + x_2^2 + x_3^2}) - (x_0 - \sqrt{x_1^2 + x_2^2 + x_3^2}) \\ &= 2\sqrt{x_1^2 + x_2^2 + x_3^2} \end{aligned} \tag{19}$$

From this, we can define a and b in terms of σ_Q:

$$
\begin{aligned}
b &= x_0 + \frac{\sigma_Q}{2} \\
a &= x_0 - \frac{\sigma_Q}{2}
\end{aligned}
\tag{20}
$$

We now have all tools required to derive a formula for hedging with σ_Q:

$$
\begin{aligned}
\nu_Q = \frac{\partial C_Q}{\partial \sigma_Q} &= \frac{\partial}{\partial \sigma_Q}\left(e^{-r\Delta t}[M_b h_b + M_a h_a]\right) \\
&= e^{-r\Delta t}\left[\frac{\partial M_b}{\partial \sigma_Q} h_b + M_b \frac{\partial h_b}{\partial \sigma_Q} + \frac{\partial M_a}{\partial \sigma_Q} h_a + M_a \frac{\partial h_a}{\partial \sigma_Q}\right] \\
&= e^{-r\Delta t}\left[\frac{h_a - h_b}{\sigma_Q^2}(e^{r\Delta t} - 1 - x_0) + \frac{S_0}{2}(M_b \Theta_b - M_a \Theta_a)\right]
\end{aligned}
\tag{21}
$$

This result provides a generalized formula for ν_Q, allowing for hedges to the most important factor in an option's price.

3.4 Quantum Rho

Rho measures the sensitivity of the option's price to changes in interest rates. This acts as a hedge against the overall economic performance of the country. To derive it, we must differentiate with respect to r.

$$
\rho_Q = \frac{\partial C_Q}{\partial r} \tag{22}
$$

r appears in both the discount factor $e^{-r\Delta t}$ and the risk neutral measures M_a and M_b:

$$
\begin{aligned}
\rho_Q &= \frac{\partial}{\partial r}\left(\frac{1}{b-a}\left[(1 - e^{-r\Delta t}(1+a))h_b + (e^{-r\Delta t}(1+b) - 1)h_a\right]\right) \\
&= \frac{1}{b-a}\left[-(-\Delta t e^{-r\Delta t}(1+a))h_b + (-\Delta t e^{-r\Delta t}(1+b))h_a\right] \\
&= \frac{\Delta t e^{-r\Delta t}}{b-a}\left[(1+a)h_b - (1+b)h_a\right]
\end{aligned}
\tag{23}
$$

3.5 Quantum Theta

The last of the Greeks derived, Θ, is the partial derivative of the option price with respect to the passage of time,

$$
\Theta_Q = -\frac{\partial C_Q}{\partial \Delta t} \tag{24}
$$

The time to expiration, Δt, appears in both the discount factor $e^{-r\Delta t}$ and within the risk-neutral measures M_a and M_b.

$$
\begin{aligned}
\Theta_Q = -\frac{\partial C_Q}{\partial \Delta t} &= -\frac{\partial}{\partial \Delta t}\left(e^{-r\Delta t}[M_b(\Delta t)h_b + M_a(\Delta t)h_a]\right) \\
&= -\left(-rC_Q + e^{-r\Delta t}\left[\frac{\partial M_b}{\partial \Delta t}h_b + \frac{\partial M_a}{\partial \Delta t}h_a\right]\right) \\
&= rC_Q - e^{-r\Delta t}\left[\left(\frac{re^{r\Delta t}}{b-a}\right)h_b - \left(\frac{re^{r\Delta t}}{b-a}\right)h_a\right] \\
&= rC_Q - \frac{r}{b-a}(h_b - h_a)
\end{aligned}
\tag{25}
$$

This shows the option's time decay is composed of two parts: growth at the risk-free rate (the rC_Q term), and a second term that depends on the interest rate and the spread of the potential option payoffs, $h_b - h_a$.

4 Simulation Methodology

To empirically test and compare the accuracy of the hedging framework for the QBOPM to that of the Classical BOPM, we create a simulation based on historical market data.

4.1 Data and Simulation Structure

- Data Sources: Our simulation uses daily closing prices for the SPDR S&P 500 ETF (ticker: SPY) as the underlying asset. For the volatility input (σ), we used the daily closing value of the CBOE Volatility Index (VIX). The risk-free rate (r) was the 13-Week Treasury Bill Rate (ÎRX). All data was sourced from Yahoo Finance for the period of January 1, 2024, to January 1, 2025.
- Simulated Option: For each trading day, we simulated the pricing and hedging of a new, at-the-money European call option with a single-day time horizon, consequently $\Delta t = \frac{1}{252}$
- Greeks: for the classical BOPM, Delta was calculated analytically while the rest were calculated numerically through the bump and revalue method. For the quantum model, the greeks were all calculated using the analytical formulae derived in Section 3.

4.2 Classical BOPM Calibration

The classical BOPM was calibrated daily using the market-implied volatility from the VIX. The parameters were determined using the Cox-Ross-Rubinstein (CRR) methodology:

1. The VIX index value was scaled by 100 to represent annualized volatility (σ)

2. The up-move (u) and down-move (d) parameters were set as: $u = e^{\sigma\sqrt{\Delta t}}$ and $d = \frac{1}{u}$

3. The risk-neutral probability was then calculated as $p = \frac{e^{r\Delta t} - d}{u - d}$

4.3 Quantum BOPM Calibration

The QBOPM has several free parameters (x_0, x_1, x_2, x_3). To ensure a well-defined daily calibration consistent with market conditions, we used a specific methodology to link these parameters to the observed VIX volatility:

1. No arbitrage condition: $\mathbb{E}[Return] = \mathrm{Tr}(\rho A) = e^{r\Delta t} - 1$

2. Calibrating volatility: $\mathrm{Var}(A) = \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho A)^2 = \sigma^2 \Delta t$

3. Parameter derivation: The market state is maximally mixed, $\rho = \frac{1}{2} I_2$, meaning it is in a state of maximum entropy. This implies that we have no prior knowledge of the market's direction, so the Bloch sphere is $(x, y, z) = (0, 0, 0)$. Additionally, the operator A is isotropic, meaning its directional components are equal ($x_1 = x_2 = x_3 = C$). From this we can derive the mean and variance:

   - $\mathrm{Tr}(\rho A) = \mathrm{Tr}(\frac{1}{2} I_2 \cdot (x_0 I_2 + C(\sigma_x + \sigma_y + \sigma_z))) = x_0 = e^{r\Delta t} - 1$.
   - $A^2 = (x_0 I_2 + C(\sigma_x + \sigma_y + \sigma_z))^2 = (x_0^2 + 3C^2) I_2 + 2x_0 C(\sigma_x + \sigma_y + \sigma_z)$

     $\mathrm{Tr}(\rho A^2) = \mathrm{Tr}\left(\frac{1}{2} I_2 \cdot A^2\right) = x_0^2 + 3C^2$. Therefore, $\mathrm{Var}(A) = (x_0^2 + 3C^2) - x_0^2 = 3C^2 \implies C = \sigma\sqrt{\frac{\Delta t}{3}}$

5 Simulation Results and Discussion

Having completed the simulation, comparing the daily outputs of both the classical and quantum model show that while the QBOPM is a novel approach to options pricing, in its current formulation it is inconsistent with foundational financial principles. This comparison is shown in Figure 1.

From this, two clear and critical observations can be made: the Quantum Vega is well above the Classical Vega at around 130, and the Quantum Theta is negative. These two distinctions highlight the places where the QBOPM falls short.

First: the Quantum Vega's extreme value is almost certainly a byproduct of the formula for Quantum Vega ($e^{-r\Delta t}\left[\frac{h_a - h_b}{\sigma_Q^2}(e^{r\Delta t} - 1 - x_0) + \frac{S_0}{2}(M_b\Theta_b - M_a\Theta_a)\right]$). This suggests the model's unreliability at shorter time frames as compared to the classical counterpart. The presence of the σ_Q^2 term in a denominator is the parameter blowing up the formula, especially because the testing was done at a daily frequency where σ_Q^2 would be very small.

Second: the most critical error found is that Quantum Theta is negative. This is a direct violation of one of the most fundamental principles in financial

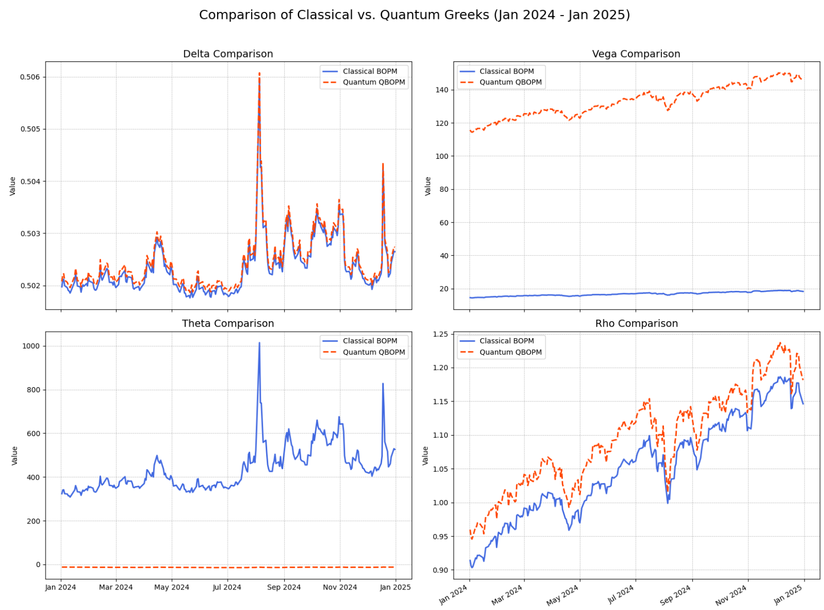

Figure 1: Comparison of Quantum vs Classical Greeks

and option pricing theory, the no-arbitrage condition. A negative theta implies that the option will gain value as time goes on, as opposed to the classical and more accurate theta decay.

6 Conclusion

The objective of this paper was to both create and numerically test hedging formulae for the Quantum Binomial Option Pricing Model. Through our analysis, we found that hedging through this model yields numerically unstable and impossible values for Quantum Greeks that betray basic postulates in options pricing and financial theory. Quantum Theta is found to be negative, which goes against the no-arbitrage condition, a foundational assumption in derivatives pricing. Additionally, Quantum Vega was blown out of scale in the tests, which was almost certainly an artifact of the small, daily frequency used in testing. This leads us to the conclusion that in its present state, the Quantum Binomial Options Pricing Model is not a viable tool to be used for hedging derivatives. However, this does not invalidate the field of quantum finance as a whole, rather it highlights the necessity of thoroughly validating and rigorously testing tools and models developed. Future work could be done in a similar fashion, by creating formulae for other options pricing models like the Black-Scholes and Heston models. Additionally, testing is still needed on creating these formulae for multi-period Quantum Binomial Options Pricing Models, which could potentially solve these errors in numerical instability.

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